Minimal Permutations and 2-Regular Skew Tableaux

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Abstract. Bouvel and Pergola introduced the notion of minimal permutations in the study of the whole genome duplication-random loss model for genome rearrangements. Let $\mathcal{F}_d(n)$ denote the set of minimal permutations of length n with d descents, and let $f_d(n) = |\mathcal{F}_d(n)|$. They derived that $f_{n-2}(n) = 2^n - (n-1)n - 2$ and $f_n(2n) = C_n$, where C_n is the n-th Catalan number. Mansour and Yan proved that $f_{n+1}(2n+1) = 2^{n-2}nC_{n+1}$. In this paper, we consider the problem of counting minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents. We show that such structures are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. Using the determinantal formula for the number of skew Young tableaux of a given shape, we find an explicit formula for $f_{n-3}(n)$. Furthermore, by using the Knuth equivalence, we give a combinatorial interpretation of a formula for a refinement of the number $f_{n+1}(2n+1)$.

Keywords: minimal permutation, 2-regular skew tableau, Knuth equivalence, the RSK algorithm.

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1 Introduction

The notion of minimal permutations was introduced by Bouvel and Pergola in the study of genome evolution, see [2]. Such permutations are a basis of permutations that can be obtained from the identity permutation via a given number of steps in the duplication-random loss model. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. A duplication of π means the duplication of a fragment of consecutive elements of π in such a way that the duplicated fragment is put immediately after the original fragment. Suppose that $\pi_i\pi_{i+1}\cdots\pi_j$ is the fragment for duplication, then the duplicated sequence is

$$\pi_1\cdots\pi_{i-1}\pi_i\cdots\pi_j\pi_i\cdots\pi_j\pi_{j+1}\cdots\pi_n.$$

A random loss means to randomly delete one occurrence of each repeated element π_k for $i \leq k \leq j$, so that we get a permutation again. In the following example, the fragment

234 is duplicated, and the underlined elements are the occurrences of repeated elements that are supposed to be deleted,

$$1\,\overbrace{234}\,56 \rightsquigarrow 1\,\overbrace{234}\,\overbrace{234}\,56 \rightsquigarrow 12\underline{3}42\underline{3}456 \rightsquigarrow 132456.$$

To describe the notation of minimal permutations, we give an overview of the descent set of a permutation and the patterns of subsequences of a permutation. Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, where $n \geq 1$. In a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, a descent is a position i such that $i \leq n-1$ and $\pi_i > \pi_{i+1}$, whereas an ascent is a position i with $i \leq n-1$ and $\pi_i < \pi_{i+1}$. For example, the permutation $3145726 \in S_7$ has two descents 1 and 5 and has four ascents $2, 3, 4$ and 6.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of distinct integers listed in increasing order, namely, $v_1 < v_2 < \cdots < v_n$. The standardization of a permutation π on V is the permutation $\mathrm{st}(\pi)$ on $[n]$ obtained from π by replacing v_i with i. For example, $\mathrm{st}(9425) = 4213$. A subsequence $\omega = \pi_{i(1)}\pi_{i(2)}\cdots\pi_{i(k)}$ of π is said to be of type σ or π contains a pattern σ if $\mathrm{st}(\omega) = \sigma$. We say that a permutation $\pi \in S_n$ contains a pattern $\tau \in S_k$ if there is a subsequence of π that is of type τ. For example, let $\pi = 263751498$. The subsequence 3549 is of type 1324, and so π contains the pattern 1324. We use the notation $\tau \prec \pi$ to denote that a permutation π contains the pattern τ, and we use $S_n(\tau_1, \ldots, \tau_k)$ to denote the set of permutations $\pi \in S_n$ that avoid the patterns $\tau_1, \tau_2, \ldots, \tau_k$.

A permutation π is called a minimal permutation with d descents if it is minimal in the sense that there exists no permutation σ with exactly d descents such that $\sigma \prec \pi$. Denote by $\mathcal{B}_d$ the set of minimal permutations with d descents. Bouvel and Pergola [2] have shown that the length, namely, the number of elements, of any minimal permutation in the set $\mathcal{B}_d$ is at least $d+1$ and at most $2d$. They also proved that in the whole genome duplication-random loss model, the permutations that can be obtained from the identity permutation in at most p steps can be characterized as permutations $d = 2^p$ descents that avoid certain patterns.

Theorem 1.1 (Bouvel and Pergola) *Let* $\pi = \pi_1\pi_2\cdots\pi_n$ *be a permutation on* $[n]$. *Then* π *is a minimal permutation with* d *descents if and only if* π *is a permutation with* d *descents satisfying the following conditions:*

(1) It starts and ends with a descent;

(2) If i is an ascent, that is, $\pi_i < \pi_{i+1}$, then $i \in \{2, 3, \ldots, n-2\}$ and $\pi_{i-1}\pi_i\pi_{i+1}\pi_{i+2}$ is of type 2143 *or* 3142.

Denote by $\mathcal{F}_d(n)$ the set of minimal permutations of length n with d descents and $f_d(n) = |\mathcal{F}_d(n)|$. Clearly, $f_d(n) = 0$ for all $d \leq 0$ or $d \geq n$, and $f_d(d+1) = 1$ for all $d \geq 1$. Bouvel and Pergola proved that $f_n(2n)$ equals the n-th Catalan number, that is,

$$f_n(2n) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

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and $f_{n-2}(n)$ is given by the formula

$$f_{n-2}(n) = 2^n - (n-1)n - 2.$$

Mansour and Yan [6] have shown that

$$f_{n+1}(2n+1) = 2^{n-2}nC_{n+1}. \tag{1.1}$$

As mentioned by Bouvel and Pergola that it is an open problem to compute $f_d(n)$ for other cases of d. In this paper, we consider the enumeration of minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents. We show that such minimal permutations are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. As a result, we may employ the determinant formula for the number of skew Young tableaux of a given shape to compute the number $f_d(n)$. With this method, we can unite the known results. Moreover, we derive an explicit formula for $f_{n-3}(n)$.

For the number $f_{n+1}(2n+1)$, we obtain a refined formula from the determinant formula. Moreover, we give a combinatorial interpretation of this formula by using the Knuth equivalence of permutations.

2 2-Regular skew tableaux

In this section, we establish a connection between the minimal permutations and skew Young tableaux of certain shape. To describe our correspondence, let us give an overview of necessary terminology on Young tableaux as used in Stanley [7].

A partition of a positive integer n is defined to be a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. If λ is a partition of n, we write $\lambda \vdash n$, or $|\lambda| = n$. The Ferrers diagram of a partition λ is a diagram with left-justified rows in which the i-th row consists of λ_i dots. The conjugate partition λ' of λ is obtained by transposing the Ferrers diagram of λ. The positive terms λ_i are called the parts of λ, and the number of parts is denoted by $l(\lambda)$.

A standard Young tableau (SYT) on $[n]$ is said to be of size n. If λ and μ are partitions with $\mu \subseteq \lambda$, namely, $\mu_i \leq \lambda_i$ for all i, we can define a standard tableau of skew shape λ/μ as a tableau on $[n]$ that is increasing in every row and every column. The number of boxes of the Young diagram of shape λ/μ is denoted by $|\lambda/\mu|$. For example, below are an SYT of shape $(4,3,3,1)$ and a skew Young tableau of shape $(6,5,2,2)/(3,1)$:

$$
\begin{array}{lll}
1 & 3 & 5 & 6 \\
2 & 4 & 8 \\
7 & 9 & 11 \\
10
\end{array}
\qquad , \qquad
\begin{array}{llll}
& & & 7 & 8 & 11 \\
& 1 & 5 & 9 & 10 \\
2 & 4 \\
3 & 6
\end{array}
\quad .
$$

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Recall that if $|\lambda/\mu| = n$ and $l(\lambda) = r$, then the number of skew Young tableaux of shape λ/μ is given by

$$f^{\lambda/\mu} = n! \det \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} \right)^r_{i,j=1},$$ (2.1)

see, for example, [7, Corollary 7.16.3].

Let $\{a_1, \ldots, a_k\}$ be a sequence of positive integers such that $a_i \geq 2$ for all i and $a_1 + a_2 + \cdots + a_k = n$. Let P be a skew Young tableau of size n with column lengths $a_1, a_2, \ldots, a_k$. We say that P is 2-regular if any two consecutive columns overlap exactly by two rows, namely, for any two consecutive columns there are exactly two rows containing elements in both columns. Denote by $\mathcal{P}_{a_1,a_2,\ldots,a_k}(n)$ the set of 2-regular skew tableaux with column lengths $a_1, a_2, \ldots, a_k$.

For example, the following skew Young tableau is 2-regular and it belongs to $\mathcal{P}_{4,2,5,3,2}(16)$:

$$
\begin{array}{cccc}
 & 6 & 8 & \\
2 & 10 & 15 & \\
5 & 11 & & \\
9 & & & \\
1 \quad 3 & 12 & & \\
4 \quad 7 & 14 & & \\
13 & & & \\
16 & & & \\
\end{array}
$$ (2.2)

For a permutation π of length n, a substring of π is a sequence of consecutive elements of π. A maximal decreasing substring of π is defined to be a decreasing substring that is not a substring of another decreasing substring. For example, the permutation $5\,2\,7\,3\,1\,4\,8\,9\,6$ contains five maximal decreasing substrings, namely, $5\,2, 7\,3\,1, 4, 8$ and $9\,6$.

It is clear that any permutation π with $k - 1$ ascents can be decomposed into k maximal decreasing substrings. To describe the ascent set, we find it convenient to use a sequence $(a_1, a_2, \ldots, a_k)$ to denote the lengths of the maximal decreasing substrings, and this sequence is called the ascent sequence of π. Then the ascent set π is expressed as $\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}\}$.

Lemma 2.1 *Given a minimal permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Suppose $(a_1, a_2, \ldots, a_k)$ is its ascent sequence, then $a_i \geq 2$ for all i.*

Proof. By condition (i) of Theorem 1.1, π starts and ends with a descent, this implies that $a_1 \geq 2$ and $a_k \geq 2$. For each ascent $j = a_1 + \cdots + a_i$ of π, where $1 \leq i \leq k - 1$, the condition (ii) of Theorem 1.1 says that $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142, which means that both $j-1$ and $j+1$ are descents. Therefore, π contains no consecutive ascents,

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and the length of decreasing sequences containing $\pi_{j-1}\pi_j$ and $\pi_{j+1}\pi_{j+2}$ are least two. This completes the proof. ∎

Let $\mathscr{F}_{a_1,a_2,\ldots,a_k}(n)$ denote the set of minimal permutations of length n with the ascent sequence $(a_1, a_2, \ldots, a_k)$, and let $F_{a_1,a_2,\ldots,a_k}(n) = |\mathscr{F}_{a_1,a_2,\ldots,a_k}(n)|$. The following theorem asserts that the number of minimal permutations with a prescribed ascent sequence is equal to the number of skew Young tableaux with fixed column lengths.

Theorem 2.2 *Let $(a_1, a_2, \ldots, a_k)$ be a sequence of positive integers such that $a_1 + a_2 \cdots + a_k = n$ and $a_i \geq 2$ for all i. Then there exists a bijection between the set $\mathscr{F}_{a_1,a_2,\ldots,a_k}(n)$ of minimal permutations with ascent sequence $(a_1, a_2, \ldots, a_k)$ and the set $\mathcal{P}_{a_1,a_2,\ldots,a_k}(n)$ of 2-regular skew tableaux with column lengths $a_1, a_2, \ldots, a_k$.*

Proof. Suppose $\pi = \pi_1\pi_2\cdots\pi_n$ is a minimal permutation in $\mathscr{F}_{a_1,a_2,\ldots,a_k}(n)$. Let $p_i = \pi_{a_i+1}\pi_{a_i+2}\cdots\pi_{a_{i+1}}$ ($0 \leq i \leq k-1$ and set $a_0 = 0$) be the k maximal decreasing substrings of π, then the elements in each p_i are strictly decreasing. Furthermore, by Theorem 1.1, if $j = a_1 + \cdots + a_i$ is an ascent then $\pi_{j-1}\pi_j\pi_{j+1}\pi_{j+2}$ is of type 2143 or 3142. Therefore, if we place these four elements into an array as follows,

$$
\begin{array}{cc}
\pi_j & \pi_{j+2} \\
\pi_{j-1} & \pi_{j+1},
\end{array}
\tag{2.3}
$$

then both its rows and columns are strictly increasing. We next construct a tableau P corresponding to π as follows. Place the elements of each maximal decreasing substring p_i in one single column, with the decreasing order from the bottom upward. This guarantees that each column of the tableau P is strictly increasing. Now, for every two adjacent maximal decreasing substrings p_i and p_{i+1}, we assume that the last two elements in p_i and the first two elements in p_{i+1} are arranged into a 2×2 square as exhibited in (2.3). This ensures that each row of P is also strictly increasing. Therefore, P is indeed a 2-regular skew tableau. To be more precise, P has the following form,

$$
P =
\begin{array}{cccc}
 & & \vdots & \\
 & & \vdots & \cdots \quad \cdots \\
 & \pi_{a_1+a_2} & \pi_{a_1+a_2+2} & \\
 & \pi_{a_1+a_2-1} & \pi_{a_1+a_2+1} & \\
 & & \vdots & \\
\pi_{a_1} & \pi_{a_1+2} & & \\
\pi_{a_1-1} & \pi_{a_1+1} & & \\
\vdots & & & \\
\pi_2 & & & \\
\pi_1 & & &
\end{array}
\tag{2.4}
$$

For example, $\pi = \underline{16}\,\underline{13}\,4\,1\,7\,3\,\underline{14}\,\underline{12}\,9\,5\,2\,\underline{11}\,\underline{10}\,6\,\underline{15}\,8 \in \mathcal{F}_{11}(16)$ contains 5 maximal decreasing substrings. The 2-regular skew tableau corresponding to π is given by the array in (2.2).

Conversely, given a 2-regular skew tableau P, we write down the elements of P from bottom up and from left to right. Then we obtain a minimal permutation π. Thus we get a bijection. ∎

For example, all the minimal permutations in $\mathcal{F}_n(2n)$ have alternating isolated descents as well as alternating isolated ascents. Note that these minimal permutations start and end with descents, see [2]. Therefore, the corresponding 2-regular skew tableaux always have straight shape (n,n),

$$
\begin{array}{ccccccc}
\pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \cdots & \pi_{2n} \\
\pi_1 & \pi_3 & \cdots & \pi_{2i-1} & \pi_{2i+1} & \cdots & \pi_{2n-1}.
\end{array}
\tag{2.5}
$$

As a result, by formula (2.1), we obtain

$$
f_n(2n) = f^{(n,n)} = (2n)! \begin{vmatrix} \dfrac{1}{n!} & \dfrac{1}{(n+1)!} \\[2mm] \dfrac{1}{(n-1)!} & \dfrac{1}{n!} \end{vmatrix}
$$

$$
= \frac{1}{n+1}\binom{2n}{n}
$$

$$
= C_n.
$$

So far we have established a one-to-one correspondence between the set $\mathscr{F}_{a_1,a_2,\ldots,a_k}$ and the set of 2-regular skew tableaux in Theorem 2.2. Hence the enumeration of the number of minimal permutations is equivalent to the enumeration of skew Young tableaux.

Corollary 2.3 *Given an ascent sequence $\alpha = (a_1, a_2, \ldots, a_k)$, where $\sum_{i=1}^{k} a_i = n$ and*

$a_i \geq 2$, for $1 \leq i \leq k$, we have

$$
F_{a_1, a_2, \ldots, a_k} = n! \det(A) = n! \begin{vmatrix} \frac{1}{a_1!} & & & & & & & \\ 1 & \frac{1}{a_2!} & & & A_{ij} & & & \\ A_{3,1} & 1 & \frac{1}{a_3!} & & & & & \\ & A_{4,2} & 1 & \frac{1}{a_4!} & & & & \\ & & \ddots & \ddots & & \ddots & & \\ & & & A_{k-1,k-3} & 1 & \frac{1}{a_{k-1}!} & & \\ & & & & A_{k,k-2} & 1 & \frac{1}{a_k!} \end{vmatrix}, \qquad (2.6)
$$

where

$$
A_{i,j} = 0, \qquad \text{when} \quad j < i - 2,
$$

$$
A_{i,i-2} = \begin{cases} 0, & \text{if} \quad a_{i-1} > 2, \\ 1, & \text{if} \quad a_{i-1} = 2, \end{cases}
$$

$$
A_{i,j} = \frac{1}{\left(\sum_{m=i}^{j} a_m - (j-i) \right)!}, \qquad \text{when} \quad j > i.
$$

Proof. First of all we need to determine the shape of the 2-regular skew tableau P defined in (2.4). Suppose the shape of P is λ/μ, by the correspondence described in Theorem 2.2, the number of elements in each column of P are $a_1, a_2, \ldots$ and a_k, respectively. In other words,

$$
\lambda_i' - \mu_i' = a_i, \quad \text{for} \quad 1 \leq i \leq k. \qquad (2.7)
$$

Furthermore, the fact that the uppermost two elements in the ith column and the lowest two elements in the $(i+1)$th column compose a 2×2 square leads to

$$
\lambda_i' - \lambda_{i+1}' = a_i - 2, \quad \text{for} \quad 1 \leq i \leq k - 1. \qquad (2.8)
$$

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Obviously $\lambda'_k = a_k$ and $\mu'_k = 0$. For $1 \leq i \leq k-1$, we have

$$\lambda'_i = \lambda'_{i+1} + (a_i - 2)$$
$$= \lambda'_{i+2} + (a_{i+1} - 2) + (a_i - 2)$$
$$= \cdots$$
$$= \lambda'_k + (a_{k-1} - 2) + \cdots + (a_i - 2)$$
$$= a_k + (a_{k-1} - 2) + \cdots + (a_i - 2)$$
$$= a_i + a_{i+1} + \cdots + a_k - 2(k - i).$$

This yields that $\mu'_i = \lambda'_i - a_i = a_{i+1} + \cdots + a_k - 2(k - i)$. So λ/μ is exactly the shape of P. As a consequence,

$$F_{a_1,a_2,\ldots,a_k}(n) = f^{\lambda'/\mu'}, \tag{2.9}$$

where

$$\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_k), \quad \lambda'_i = \sum_{j=i}^{k} a_j - 2(k - i), \quad \text{for} \quad 1 \leq i \leq k,$$

and

$$\mu' = (\mu'_1, \mu'_2 \ldots, \mu'_k), \quad \mu'_i = \lambda'_i - a_i = \sum_{j=i+1}^{k} a_j - 2(k - i), \quad \text{for} \quad 1 \leq i \leq k.$$

Now we proceed to compute the number of 2-regular skew tableaux by using formula (2.1). We consider the shape λ'/μ' of the 2-regular skew tableau P by dividing the relations between i and j into the following cases. From the equations (2.7) and (2.8), we obtain

(1). If $j < i - 2$, $\lambda'_i - \mu'_j = -(a_{j+1} + \cdots + a_{i-1}) + 2(i - j)$. Therefore, $\lambda'_i - \mu'_j - i + j = -(a_{j+1} + \cdots + a_{i-1}) + (i - j)$. Since $a_m \geq 2 (1 \leq m \leq k)$, we see that $\lambda'_i - \mu'_j - i + j < 0$, which means $A_{i,j} = 0$.

(2). If $j = i - 2$, $\lambda'_i - \mu'_{i-2} = -a_{i-1} + 4$. Therefore, if $a_{i-1} = 2$, then $\lambda'_i - \mu'_{i-2} - i + i - 2 = 0$ and $A_{i,i-2} = 1/0! = 1$. Otherwise, we have $a_{i-1} > 2$, $\lambda'_i - \mu'_{i-2} - i + i - 2 < 0$, and $A_{i,i-2} = 0$.

(3). If $j = i - 1$, $\lambda'_i - \mu'_{i-1} = 2$. Consequently, $\lambda'_i - \mu'_{i-1} - i + (i - 1) = 1$, and $A_{i,i-1} = 1$.

(4). If $j = i$, then from the relation (2.7), we have $\lambda'_i - \mu'_i = a_i$, and so $A_{i,i} = \dfrac{1}{a_i!}$.

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(5). If $j \geq i-1$, then by (2.8), $\lambda_i' - \mu_j' = a_i + (a_{i+1} - 2) + \cdots + (a_j - 2) = a_i + \cdots a_j - 2(j-i)$. In this case,

$$A_{i,j} = \frac{1}{(\lambda_i' - \mu_j' - i + j)!} = \frac{1}{(a_i + \cdots + a_j - (j-i))!},$$

as desired. This completes the proof. ∎

We remark that Corollary 2.3 can be viewed as a refinement of the number $f_d(n)$, that is the number of minimal permutations in $\mathcal{F}_{n-k}(n)$ with prescribed ascent set $\{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{k-1}\}$.

We can now compute the number of minimal permutations in $\mathcal{F}_d(n)$. Note that such minimal permutations have $n - d$ maximal decreasing substrings.

Corollary 2.4 *For $d + 1 \leq n \leq 2d$, we have*

$$f_d(n) = \sum_{\substack{a_i \geq 2 \, for \, 1 \leq i \leq n-d \\ a_1 + a_2 + \cdots + a_{n-d} = n}} F_{a_1, a_2, \ldots, a_{n-d}}. \tag{2.10}$$

As an application of the above formula (2.10) for $d = n - 2$, we immediately obtain the formula for $f_{n-2}(n)$ due to Bouvel and Pergola [2]. It is obvious that the minimal permutations in $\mathcal{F}_{n-2}(n)$ have only one ascent, which implies that they have two maximal decreasing substrings. Suppose that the unique ascent is k, and the ascent sequence is $(k, n - k)$ for $2 \leq k \leq n - 2$. By Corollary 2.3, it is easy to check

$$F_{k,n-k} = n! \begin{vmatrix} \dfrac{1}{k!} & \dfrac{1}{(n-1)!} \\ 1 & \dfrac{1}{(n-k)!} \end{vmatrix} = \binom{n}{k} - n.$$

By Corollary 2.4, we arrive at

$$f_{n-2}(n) = \sum_{k=2}^{n-2} \left(\binom{n}{k} - n \right) = 2^n - 2 - n(n-1).$$

We now come to the computation of $f_{n-3}(n)$.

Theorem 2.5 *The number of minimal permutations of length n with $n-3$ descents equals*

$$f_{n-3}(n) = 3^n - (n^2 - 2n + 4)2^{n-1} + \frac{1}{2}\left(n^4 - 7n^3 + 19n^2 - 21n + 2\right).$$

Proof. The minimal permutations in $\mathcal{F}_{n-3}(n)$ have three maximal decreasing substrings. Suppose the ascent sequence is (a, b, c) such that $a + b + c = n$ and $a, b, c \geq 2$. According to Corollary 2.3, let

$$
A_1 = n! \begin{vmatrix} \dfrac{1}{a!} & \dfrac{1}{(a+1)!} & \dfrac{1}{(n-2)!} \\[2mm] 1 & \dfrac{1}{2!} & \dfrac{1}{(c+1)!} \\[2mm] 1 & 1 & \dfrac{1}{c!} \end{vmatrix}
$$

$$
= \frac{n!}{a!2!c!} + \frac{n!}{(a+1)!(c+1)!} + \frac{n!}{(n-2)!}
$$

$$
- \frac{n!}{(n-2)!2!} - \frac{n!}{(a+1)!c!} - \frac{n!}{a!(c+1)!},
$$

and let

$$
A_2 = n! \begin{vmatrix} \dfrac{1}{a!} & \dfrac{1}{(a+b-1)!} & \dfrac{1}{(n-2)!} \\[2mm] 1 & \dfrac{1}{b!} & \dfrac{1}{(b+c-1)!} \\[2mm] 0 & 1 & \dfrac{1}{c!} \end{vmatrix}
$$

$$
= \frac{n!}{a!b!c!} + \frac{n!}{(n-2)!} - \frac{n!}{(a+b-1)!c!} - \frac{n!}{a!(b+c-1)!}.
$$

By Corollary 2.4, we obtain

$$
f_{n-3}(n) = \sum_{\substack{a,c \geq 2, b=2 \\ a+b+c=n}} A_1 + \sum_{\substack{a,c \geq 2, b \geq 3 \\ a+b+c=n}} A_2.
$$

In order to simplify the computation, we reformulate the above equation into the following form,

$$
f_{n-3}(n) = \sum_{\substack{a,c \geq 2, b=2 \\ a+b+c=n}} A_1 + \sum_{\substack{a,b,c \geq 2 \\ a+b+c=n}} A_2 - \sum_{\substack{a,c \geq 2, b=2 \\ a+b+c=n}} A_2
$$

$$
= \sum_{\substack{a,c \geq 2, b=2 \\ a+b+c=n}} (A_1 - A_2) + \sum_{\substack{a,b,c \geq 2 \\ a+b+c=n}} A_2. \tag{2.11}
$$

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It is easy to check that

$$\sum_{\substack{a,c\geq 2,b=2 \\ a+b+c=n}} (A_1 - A_2) = \sum_{\substack{a,c\geq 2 \\ a+c=n-2}} \left(\frac{n!}{(a+1)!(c+1)!} - \binom{n}{2} \right)$$

$$= \sum_{a=2}^{n-4} \left(\binom{n}{a+1} - \binom{n}{2} \right)$$

$$= 2^n - 2n - 2 - (n-3)\binom{n}{2}. \qquad (2.12)$$

The second sum of (2.11) can be expressed as follows

$$\sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} A_2 = \sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} \left(\binom{n}{a,b,c} + n(n-1) - n\binom{n-1}{c} - n\binom{n-1}{a} \right).$$

On the one hand, by the inclusion-exclusion principle, we have

$$\sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} \binom{n}{a,b,c} = 3^n - 3\sum_{b=0}^{n} \binom{n}{0,b,n-b} - 3\sum_{b=0}^{n-1} \binom{n}{1,b,n-1-b}$$

$$+ 3\binom{n}{0,0,n} + 3\binom{n}{1,1,n-2} + 6\binom{n}{0,1,n-1}$$

$$= 3^n - 3\cdot 2^n - 3n\cdot 2^{n-1} + 3n^2 + 3n + 3. \qquad (2.13)$$

On the other hand, let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$, then we get

$$\sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} n(n-1) = n(n-1) \cdot [x^n](x^2 + x^3 + \cdots)^3$$

$$= n(n-1)\binom{n-4}{2}. \qquad (2.14)$$

Furthermore,

$$n \sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} \left(\binom{n-1}{c} + \binom{n-1}{a} \right) = 2n \sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} \binom{n-1}{a}$$

It is easily seen that

$$\sum_{\substack{a,b,c\geq 2 \\ a+b+c=n}} \binom{n-1}{a} = \sum_{a=2}^{n-4} \sum_{\substack{b,c\geq 2 \\ b+c=n-a}} \binom{n-1}{a} = \sum_{a=2}^{n-4} \binom{n-1}{a}(n-3-a).$$

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Since

$$\sum_{a=1}^{n-1} a \binom{n-1}{a} = \sum_{a=1}^{n-1} a \binom{n-1}{a} x^{a-1} \bigg|_{x=1} = (n-1)(1+x)^{n-2} \bigg|_{x=1} = (n-1)2^{n-2},$$

we find

$$2n \sum_{\substack{a,b,c \geq 2 \\ a+b+c=n}} \binom{n-1}{a} = 2n(n-3) \sum_{a=2}^{n-4} \binom{n-1}{a} - 2n \sum_{a=2}^{n-4} a \binom{n-1}{a}$$

$$= 2n(n-3) \left(2^{n-1} - 2(n-1) - 2 \right)$$

$$\qquad - 2n \left((n-1)2^{n-2} - 2(n-1) - (n-1)(n-2) \right)$$

$$= n(n-3)2^n - n(n-1)2^{n-1} - 2n^3 + 10n^2. \tag{2.15}$$

By (2.12), (2.13), (2.14) and (2.15), we finally obtain

$$f_{n-3}(n) = 3^n - (n^2 - 2n + 4)2^{n-1} + \frac{1}{2} \left(n^4 - 7n^3 + 19n^2 - 21n + 2 \right),$$

as claimed. ∎

3 A refinement of $f_{n+1}(2n+1)$ via Knuth equivalence

In this section, we give a combinatorial proof of a refined formula for the number $f_{n+1}(2n+1)$. Given a minimal permutation $\pi = \pi_1 \pi_2 \cdots \pi_{2n+1}$ of length $2n+1$ with $n+1$ descents, there are only one occurrence of consecutive descents in π and the other descents are separated by ascents. We shall consider the set of minimal permutations for which the unique consecutive descents are $2i-1$ and $2i$.

Theorem 3.1 *Let $\mathcal{M}_{2n+1,2i}$ be the subset of $\mathcal{F}_{n+1}(2n+1)$ whose unique consecutive descents are $2i-1$ and $2i$, where $1 \leq i \leq n$. We have*

$$|\mathcal{M}_{2n+1,2i}| = \binom{2n+1}{n-1}\binom{n-1}{i-1}. \tag{3.1}$$

We shall give two proofs of this theorem.

It is easy to see that the 2-regular skew tableaux corresponding to minimal permutations $\pi \in \mathcal{M}_{2n+1,2i}$ are of the following form,

$$
\begin{array}{ccccccccc}
 & & & \pi_{2i+1} & \pi_{2i+3} & \pi_{2i+5} & \cdots & \pi_{2n-1} & \pi_{2n+1} \\
\pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \pi_{2i+4} & \cdots & \pi_{2n-2} & \pi_{2n} \\
\pi_1 & \pi_3 & \cdots & \pi_{2i-1}. & & & & &
\end{array} \tag{3.2}
$$

The conjugate shape λ'/μ' is

$$(\underbrace{3,3,\ldots,3}_{i},\underbrace{2,2,\ldots,2}_{n-i})/(\underbrace{1,1,\ldots,1}_{i-1}), \quad \text{for} \quad 1 \leq i \leq n.$$

Notice that the skew shape λ/μ is $(n,n,i)/(i-1)$. In this context, we can obtain the number of skew Young tableaux directly from formula 2.1,

$$f^{(n,n,i)/(i-1)} = (2n+1)! \begin{vmatrix} \dfrac{1}{(n-i+1)!} & \dfrac{1}{(n+1)!} & \dfrac{1}{(n+2)!} \\[2mm] \dfrac{1}{(n-i)!} & \dfrac{1}{n!} & \dfrac{1}{(n+1)!} \\[2mm] 0 & \dfrac{1}{(i-1)!} & \dfrac{1}{i!} \end{vmatrix}$$

$$= \binom{2n+1}{n}\binom{n+1}{i} + \binom{2n+1}{n-1}\binom{n-1}{i-1}$$

$$- \binom{2n+1}{n}\binom{n}{i} - \binom{2n+1}{n}\binom{n}{i-1}$$

$$= \binom{2n+1}{n-1}\binom{n-1}{i-1}.$$

So we immediately get

$$f_{n+1}(2n+1) = \sum_{i=1}^{n} f^{(n,n,i)/(i-1)} = 2^{n-1}\binom{2n+1}{n-1}.$$

∎

We next give a combinatorial interpretation of Theorem 3.1. We give an overview of the background on the RSK correspondence and the Knuth equivalence.

Suppose $\pi \xrightarrow{\text{RSK}} (P,Q)$, where P is called the insertion tableau while Q is called the recording tableau. Two permutations are Knuth-equivalent if and only if their insertion tableaux are the same.

The following properties of insertion paths will be useful. Denote by $I(P \leftarrow k)$ the insertion path of a positive integer k into an SYT (standard Young tableau) $P = (P_{ij})$ by the RSK algorithm. Then

(a) When we insert k into an SYT P, the insertion path moves to the left. More precisely, if $(r,s), (r+1,t) \in I(P \leftarrow k)$ then $t \leq s$.

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(b) Let P be an SYT, and let $j < k$. Then $I(P \leftarrow j)$ lies strictly to the left of $I((P \leftarrow j) \leftarrow k)$. More precisely, if $(r, s) \in I(P \leftarrow j)$, and $(r, t) \in I((P \leftarrow j) \leftarrow k)$, then $s < t$. Moreover, $I((P \leftarrow j) \leftarrow k)$ does not extend below the bottom of $I(P \leftarrow j)$. Equivalently,

$$\#I((P \leftarrow j) \leftarrow k) \leq \#I(P \leftarrow j).$$

See [7] for more details.

Now we begin the combinatorial proof by using the Knuth equivalence of permutations in connection with the RSK correspondence between permutations and standard Young tableaux, see Stanley [7].

Let $\mathcal{T}_{2n+1,k}$ be the set of standard Young tableaux of size $2n + 1$ with shape $(n, n + 1 - k, k)$, where $1 \leq k \leq [\frac{n+1}{2}]$, where $[x]$ denotes the largest integer not exceeding x. According to the correspondence between minimal permutations and 2-regular skew tableaux, we see that $\mathcal{M}_{2n+1,2i}$ is the set of the 2-regular skew tableaux with row lengths $n - i + 1, n, i$.

Theorem 3.2 *There exists a bijection between $\mathcal{M}_{2n+1,2i}$ and $\mathcal{T}_{2n+1,k}$ such that the length of the last row of an SYT in $\mathcal{T}_{2n+1,k}$ does not exceed the smallest row length of the corresponding 2-regular tableau in $\mathcal{M}_{2n+1,2i}$. Equivalently, we have the following formula,*

$$|\mathcal{M}_{2n+1,2i}| = \sum_{k=1}^{j} |\mathcal{T}_{2n+1,k}|, \qquad where \quad j = \min\{n - i + 1, i\}. \tag{3.3}$$

Consequently, $|\mathcal{M}_{2n+1,2i}|$ is symmetric in i,

$$|\mathcal{M}_{2n+1,2i}| = |\mathcal{M}_{2n+1,2(n-i+1)}|. \tag{3.4}$$

The main idea of the proof can be described as follows. For every $\pi \in \mathcal{M}_{2n+1,2i}$, we show that there always exists a permutation $\pi' \in \mathfrak{S}_{2n+1}$ which equivalent to π (by the Knuth equivalence, to be precise). In other words, they have the same insertion tableau. Therefore, by constructing the insertion tableau of π', we can give a description of the insertion tableau of π. Thus we obtain the shapes of the SYTs corresponding to the permutations in $\mathcal{M}_{2n+1,2i}$.

Proof. Let $\pi' = \pi_1\pi_2 \cdots \pi_{2i-1}\pi_{2i}\pi_{2i+2} \cdots \pi_{2n}\pi_{2i+1}\pi_{2i+3} \cdots \pi_{2n+1} \in \mathfrak{S}_{2n+1}$, that is, π' is obtained from π by fixing the first $2i$ elements, and moving the remaining elements with even subscripts forward and those with odd subscripts backward. By (3.2), π' can also obtained by first reading the elements of the last two rows of (3.2) and keeping the order of these elements in π unchanged. Then read off the elements of the first row of (3.2).

First, we show that π and π' are Knuth equivalent, namely,

$$\pi \overset{K}{\sim} \pi'.$$

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Recall that each Knuth transformation switches two adjacent entries a and c provided that an entry b satisfying $a < b < c$ is located next to a or c. Write

$$\pi = \pi_1\pi_2\cdots\pi_{2i-1}\pi_{2i}\,\boxed{\pi_{2i+1}}\,\pi_{2i+2}\,\boxed{\pi_{2i+3}}\,\cdots\pi_{2n}\,\boxed{\pi_{2n+1}}.$$

For the purpose of presentation, the elements which will be moved back are framed. When we write bac under three consecutive elements, we mean that these three elements have type bac. We shall apply a series of Knuth transformations to the substring $\pi_{2i}\,\boxed{\pi_{2i+1}}\,\pi_{2i+2}\cdots\boxed{\pi_{2n+1}}$ of π. This is equivalent to exchanging every two adjacent elements after π_{2i},

$$\pi = \pi_1\cdots\pi_{2i-1}\,\pi_{2i}\underbrace{\boxed{\pi_{2i+1}}\,\pi_{2i+2}}_{bac}\,\underbrace{\boxed{\pi_{2i+3}}\,\pi_{2i+4}\,\boxed{\pi_{2i+5}}}_{acb}\,\pi_{2i+6}\cdots\boxed{\pi_{2n-1}}\,\pi_{2n}\,\boxed{\pi_{2n+1}}$$

$$\overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i-1}\pi_{2i}\pi_{2i+2}\,\boxed{\pi_{2i+1}}\,\boxed{\pi_{2i+4}}\,\pi_{2i+3}\,\underbrace{\boxed{\pi_{2i+5}}\,\pi_{2i+6}\,\boxed{\pi_{2i+7}}}_{acb}\,\pi_{2i+8}\,\boxed{\pi_{2i+9}}\,\pi_{2i+10}\,\boxed{\pi_{2i+11}}\cdots$$

$$\cdots\cdots$$

$$\overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i-1}\pi_{2i}\underbrace{\pi_{2i+2}\,\boxed{\pi_{2i+1}}\,\pi_{2i+4}}_{bac}\,\underbrace{\boxed{\pi_{2i+3}}\,\pi_{2i+6}\,\boxed{\pi_{2i+5}}}_{acb}\,\pi_{2i+8}\,\boxed{\pi_{2i+9}}\cdots\pi_{2n}\,\boxed{\pi_{2n-1}}\,\boxed{\pi_{2n+1}}.$$

By this procedure we have moved π_{2i+2} forward and π_{2n-1} backward, Then for the substring $\pi_{2i+2}\,\boxed{\pi_{2i+1}}\,\pi_{2i+4}\cdots\boxed{\pi_{2n+1}}$ of the resulting permutation, repeat the above procedure in order to move π_{2i+4} forward and π_{2n-3} backward.

$$\pi \overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i}\pi_{2i+2}\pi_{2i+4}\,\boxed{\pi_{2i+1}}\,\boxed{\pi_{2i+6}}\,\pi_{2i+3}\,\underbrace{\boxed{\pi_{2i+5}}\,\pi_{2i+8}\,\boxed{\pi_{2i+7}}}_{acb}\,\pi_{2i+10}\,\boxed{\pi_{2i+9}}\cdots$$

$$\overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i}\pi_{2i+2}\pi_{2i+4}\,\boxed{\pi_{2i+1}}\,\boxed{\pi_{2i+6}}\,\pi_{2i+3}\,\boxed{\pi_{2i+8}}\,\pi_{2i+5}\,\underbrace{\boxed{\pi_{2i+7}}\,\pi_{2i+10}\,\boxed{\pi_{2i+9}}}_{acb}\cdots$$

$$\cdots\cdots$$

$$\overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i}\pi_{2i+2}\underbrace{\pi_{2i+4}\,\boxed{\pi_{2i+1}}\,\pi_{2i+6}}_{bac}\,\underbrace{\boxed{\pi_{2i+3}}\,\pi_{2i+8}\,\boxed{\pi_{2i+5}}}_{acb}\,\pi_{2i+10}\cdots\pi_{2n}\,\boxed{\pi_{2n-3}}\,\boxed{\pi_{2n-1}}\,\boxed{\pi_{2n+1}}.$$

Iterating this process until all the elements after π_{2i} with even subscripts are moved forward while the elements after π_{2i} of odd subscripts are moved backward, we get

$$\pi \overset{\mathrm{K}}{\sim}\pi_1\cdots\pi_{2i-1}\pi_{2i}\pi_{2i+2}\pi_{2i+4}\cdots\pi_{2n-2}\pi_{2n}\,\boxed{\pi_{2i+1}}\,\boxed{\pi_{2i+3}}\,\boxed{\pi_{2i+5}}\cdots\boxed{\pi_{2n-3}}\,\boxed{\pi_{2n-1}}\,\boxed{\pi_{2n+1}}$$

$$= \pi'.$$

We give an example to illustrate the above procedure. Let $\pi = 6\,3\,7\,4\,1\,5\,2\,9\,8\,11\,10\,13\,12 \in \mathcal{F}_7(13)$. We have the following Knuth transformations:

$$\pi = 6\,3\,7\,4\,\boxed{1}\,5\,\boxed{2}\,9\,\boxed{8}\,11\,\boxed{10}\,13\,\boxed{12} \overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,\boxed{1}\,9\,\boxed{2}\,\boxed{8}\,11\,\boxed{10}\,13\,\boxed{12}$$

$$\overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,\boxed{1}\,9\,\boxed{2}\,11\,\boxed{8}\,\boxed{10}\,13\,\boxed{12} \overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,\boxed{1}\,9\,\boxed{2}\,11\,\boxed{8}\,13\,\boxed{10}\,\boxed{12}$$

$$\overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,9\,\boxed{1}\,11\,\boxed{2}\,\boxed{8}\,13\,\boxed{10}\,\boxed{12} \overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,9\,\boxed{1}\,11\,\boxed{2}\,13\,\boxed{8}\,\boxed{10}\,\boxed{12}$$

$$\overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,9\,11\,\boxed{1}\,\boxed{2}\,13\,\boxed{8}\,\boxed{10}\,\boxed{12} \overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,9\,11\,\boxed{1}\,13\,\boxed{2}\,\boxed{8}\,\boxed{10}\,\boxed{12}$$

$$\overset{\mathrm{K}}{\sim} 6\,3\,7\,4\,5\,9\,11\,13\,\boxed{1}\,\boxed{2}\,\boxed{8}\,\boxed{10}\,\boxed{12}$$

$$= \pi'.$$

Therefore, π and π' have the same insertion tableau. Applying the RSK algorithm to π', it is easy to see that the standard Young tableau corresponding to the first $n+i$ elements $\pi_1\pi_2\cdots\pi_{2i}\pi_{2i+2}\cdots\pi_{2n}$ of π' is an SYT of shape (n,i),

$$P' = \begin{matrix} \pi_2 & \pi_4 & \cdots & \pi_{2i} & \pi_{2i+2} & \cdots & \pi_{2n} \\ \pi_1 & \pi_3 & \cdots & \pi_{2i-1}, & & & \end{matrix} \tag{3.5}$$

which is exactly the last two rows of (3.2). The insertion tableau of π' can be obtained as follows

$$\left(\left(\cdots ((P' \leftarrow \pi_{2i+1}) \leftarrow \pi_{2i+3}) \cdots \leftarrow \right) \pi_{2n-1} \right) \leftarrow \pi_{2n+1}. \tag{3.6}$$

Since π and π' have the same insertion tableau, (3.6) can be also considered as the insertion tableau of π.

We now aim to give a second combinatorial proof. First we show that

$$\mathcal{M}_{2n+1,2i} \longrightarrow \bigcup_{k=1}^{j} \mathcal{T}_{2n+1,k}$$

is an injection. Since $\pi_{2i+1} > \pi_{2i} > \pi_{2i-1}$, when inserting π_{2i+1} into $P' = (P_{ij})$, the resulting tableau corresponding $P_0 = P' \leftarrow \pi_{2i+1}$ is of shape $(n,i,1)$. Moreover, the intersection position of $I(P_0)$ in the first row cannot be to the right of $(i,1)$ in P'. By induction, we assume that the insertion tableau of $P_{m-1} = (\cdots(P' \leftarrow \pi_{2i+1})\cdots \leftarrow \pi_{2i+2(m-1)+1})$ is of shape $(n, i+m-s, s)$, where $1 \le s \le m$. Then let us examine the shape of $P_m = P_{m-1} \leftarrow \pi_{2i+2m+1}$. Since $\pi_{2i+2m+1} > \pi_{2i+2m-1}$, the insertion path $I(P_m)$ lies strictly to the right of $I(P_{m-1})$ and does not extend below the bottom of $I(P_{m-1})$. Since $\pi_{2i+2m+1} > \pi_{2i+2m}$, the insertion path of P_m in the first row cannot extend to the right of

$(i+m, 1)$ of P'. It follows that the shape of P_m can be obtained from that of P_{m-1} by adding a new element to the second or the third row of P_{m-1}. We deduce that the shape of P_m must be one of the form $(n, i+m+1-s, s)$, where $1 \leq s \leq m+1$.

Next we aim to show that

$$\bigcup_{k=1}^{j} \mathcal{T}_{2n+1,k} \longrightarrow \mathcal{M}_{2n+1,2i}$$

is also an injection. Given a standard Young tableau P of shape $(n, n+1-k, k)$. Pick up the set of positions in P which are note occupied by elements in P' given by (3.5). Suppose that these positions are $(i_1, j_1), (i_2, j_2), \ldots, (i_{n-i}, j_{n-i})$ such that $j_1 \geq j_2 \cdots \geq j_{n-i}$ and that if $j_t = j_{t+1}$, then $i_t < i_{t+1}$. In other words, these positions are ordered from the northeast corner to the southwest corner.

At first, we apply the "inverse bumping" to $P_{i_1 j_1}$. It bumps an element $\pi_{r_1 t_1}$ in the first row from P. Put $\pi_{r_1 t_1}$ on top of π_{1n} of $P \to P_{i_1 j_1}$. It is easy to see that $\pi_{r_1 t_1} < \pi_{1n}$. Note that when we begin to apply the "inverse bumping" to $P_{i_1 j_1}$, it is put the end of its row (row i_1). Inductively, suppose $P_{i_s j_s}$ bumps some element $\pi_{r_s t_s}$ in the first row of P, and $\pi_{r_s t_s}$ is putted on top of $\pi_{1,n-s+1}$, where $\pi_{r_s t_s} < \pi_{1,n-s+1}$. When we apply the inverse bumping to $P_{i_{s+1} j_{s+1}}$, its "inverse insertion path" intersecting row i_s is strictly to the left of column j_s. Consequently, at row i_s, the inverse insertion path of $P_{i_{s+1} j_{s+1}}$ lies strictly to the left of that of $P_{i_s j_s}$. By induction, the entire inverse insertion path of $P_{i_{s+1} j_{s+1}}$ lies strictly to the left of that of $P_{i_s j_s}$. In particular, the element $\pi_{r_{s+1} t_{s+1}}$ bumped by $P_{i_{s+1} j_{s+1}}$ in the first row is to the left of $\pi_{r_s t_s}$. Hence $\pi_{r_{s+1} t_{s+1}} < \pi_{r_s t_s}$.

We now put $\pi_{r_{s+1} t_{s+1}}$ to the left of $\pi_{r_s t_s}$. Since the inverse insertion path of $P_{i_{s+1} j_{s+1}}$ lies strictly to the left of that of $P_{i_s j_s}$, we find $\pi_{r_{s+1} t_{s+1}} < \pi_{1\,n-s}$, as required. Note that the condition $j = \min i, n+1-i$ is necessary since the resulting tableau is a standard Young tableau.

The symmetry of $\mathcal{M}_{2n+1,2i}$ is immediate from (3.3). ∎

We now aim to compute the number of SYTs of shape $(n, n+1-k, k)$. Recall that if $\lambda \vdash n$, then the number of SYTs of shape λ is given by the hook length formula,

$$f^{\lambda} = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u) = \lambda_i + \lambda'_j - i - j + 1$.

By the hook length formula, it is easy to show that $\binom{2n+1}{n-1}$ counts the number of SYTs of shape $(n, n, 1)$. For the general case, we have

Theorem 3.3 *For $2 \leq k \leq [\frac{n+1}{2}]$, the number of SYTs of shape $(n, n+1-k, k)$ is*

$$|\mathcal{T}_{2n+1,k}| = \frac{n-2k+2}{k-1}\binom{n-1}{k-2}\binom{2n+1}{n-1}. \tag{3.7}$$

Proof. We conduct induction on k. When $k = 2$, by the hook length formula, it is easy to show that the number of SYTs of shape $(n, n - 1, 2)$ equals

$$(n - 2)\binom{2n + 1}{n - 1}.$$

We now suppose that (3.7) holds for $k - 1$. Comparing the hook lengths of SYTs of shape $(n, n - k + 1, k)$ to those of shape $(n, n + 2 - k, k - 1)$, we find that

$$\frac{|\mathcal{T}_{2n+1,k-1}|}{|\mathcal{T}_{2n+1,k}|} = \frac{(n - 2k + 2)(n - k + 2)}{(n - 2k + 4)(k - 1)}.$$

Thus the number of SYTs of shape $(n, n - k + 1, k)$ is given by

$$|\mathcal{T}_{2n+1,k}| = \frac{(n - 2k + 2)(n - k + 2)}{(n - 2k + 4)(k - 1)} \frac{(n - 2k + 4)}{k - 2} \binom{n - 1}{k - 3}\binom{2n + 1}{n - 1}$$

$$= \frac{n - 2k + 2}{k - 1}\binom{n - 1}{k - 2}\binom{2n + 1}{n - 1}.$$

∎

We are now ready to complete the proof of Theorem 3.1.

The proof of Theorem 3.1. We use induction on i. By the symmetry of $|\mathcal{M}_{2n+1,2i}|$ with respect to i, it suffices to consider the case $i \leq (n + 1)/2$. Note that the theorem holds for $i = 1$. Suppose that

$$|\mathcal{M}_{2n+1,2(i-1)}| = \binom{n - 1}{i - 2}\binom{2n + 1}{n - 1}.$$

By Theorems 3.2 3.3, we deduce that

$$|\mathcal{M}_{2n+1,2i}| = |\mathcal{M}_{2n+1,2(i-1)}| + |\mathcal{T}_{2n+1,i}|$$

$$= \left(\binom{n - 1}{i - 2} + \frac{n - 2i + 2}{i - 1}\binom{n - 1}{i - 2}\right)\binom{2n + 1}{n - 1}$$

$$= \binom{n - 1}{i - 1}\binom{2n + 1}{n - 1},$$

as desired. This completes the proof. ∎

References

[1] M.H. Albert, R.E.L. Aldred, M.D. Atkinson, H.P. Van Ditmarsch, C.C. Handley, D.A. Hotlon and D.J. McCaughan, Compositions of pattern restricted sets of permutations, Technical report, OUCS-2004-12, University of Otago, 2004.

[2] M. Bouvel and E. Pergola, Posets and permutations in the duplication-loss model: minimal permutations with d descents, Theoret. Comput. Sci., 411 (2010), 2487–2501.

[3] S. Bérard, A. Bergeron, C. Chauve and C. Paul, Perfect sorting by reversals is not always difficult, IEEE/ACM Trans. Comput. Biol. Bioinformatics, 4 (2007), 4–6.

[4] K. Chaudhuri, K. Chen, R. Mihaescu and S. Rao, On the tandem duplication-random loss model of genome rearrangement, Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm, ACM, New York, 2006, pp. 564–570.

[5] A. Labarre, New bounds and tractable instances for the transposition distance, IEEE/ACM Trans. Comput. Biology Bioinform, 3 (2006), 380–394.

[6] T. Mansour and S.H.F. Yan, Minimal permutations with d descents, European J. Combin., 31 (2010), 1445–1460.

[7] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.

[8] R.P. Stanley, Polygon dissections and standard Young tableaux, J. Combin. Theory Ser. A, 76 (1996), 175–176.

[9] R.P. Stanley, On the enumeration of skew Young tableaux, Adv. Appl. Math., 30 (2003), 283–294.